Sur l'équivalence et la stabilité des systèmes multidimensionnels linéaires

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Issu de différentes collaborations avec *O. Bachelier, N. Yeganefar, R. David* (Univ. Poitiers - LIAS), *A. Quadrat* (Inria Paris) et *F. J. Silva Alvares* (Univ. Limoges - XLIM)

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Starting point

◇ State-space rep. of a 2D linear discrete Roesser model (Roesser'75):

$$(\mathrm{R}) \begin{cases} \begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + \underbrace{\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}}_{B} u(i,j) \\ y(i,j) = \underbrace{\begin{pmatrix} C_1 & C_2 \end{pmatrix}}_{C} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + D\, u(i,j) \end{cases}$$

◇ We dispose of techniques to study stability and stabilizations issues for this particular type of models (*Bachelier et al*)

◇ **Idea**: use equivalence transformations to study stability and stabilizations issues for other classes of models:

- Starting from a model, find an equivalent Roesser model
- Apply known techniques to the Roesser model
- Deduce properties of the initial model

Content of the presentation

1. Algebraic analysis approach to linear systems theory

   $\rightarrow$ construct systems equivalence via matrix computations

2. Application to study stability and stabilizations issues in multidimensional (nD) systems theory (in this talk $n = 2$)

   $\rightarrow$ reduce certain classes of nD systems (in this talk, linear repetitive processes) to equivalent Roesser models

   $\rightarrow$ use existing techniques specific to Roesser models to study stability and stabilizations issues

Algebraic Analysis Approach to Linear Systems Theory: Methodology

1. A linear system is defined by a matrix R with coefficients in a ring D of functional operators:

$$R\,y = 0 \quad (\star)$$

Roesser Model

$$(\mathrm{R}):\ \left\{\begin{array}{lcl} r(i+1,j) & = & 2\,r(i,j)+3\,s(i,j) \\ s(i,j+1) & = & 4\,r(i,j)+5\,s(i,j) \end{array}\right.$$

$\diamond$ $D=\mathbb{R}[\sigma_i,\sigma_j]$ ring of partial shift operators with constant coeff. in $\mathbb{R}$:

$$\delta \in D,\ \delta = \sum_{k,l} \underbrace{d_{kl}}_{\in\mathbb{R}} \sigma_i^k\,\sigma_j^l, \quad \delta\, u(i,j) = \sum_{k,l} d_{kl}\, u(i+k,j+l)$$

$\diamond$ The (R) model can then be written $R\,y=0$ with:

$$R = \begin{pmatrix} \sigma_i-2 & -3 \\ -4 & \sigma_j-5 \end{pmatrix} \in D^{2\times 2}, \quad y(i,j) = \begin{pmatrix} r(i,j) \\ s(i,j) \end{pmatrix}$$

Algebraic Analysis Approach to Linear Systems Theory: Methodology

1. A linear system is defined by a matrix R with coefficients in a ring D of functional operators:

$$R\,y = 0 \quad (\star)$$

2. To $(\star)$ we associate a left D-module M (finitely presented)

The left D-module M

◇ D ring of functional operators, $R \in D^{q \times p}$

◇ To the matrix R we associate the left D-module:

$$M = D^{1 \times p} / (D^{1 \times q} \, R)$$

given by the following finite presentation

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda \, R & & & & \\ & & \delta & \longmapsto & \pi(\delta) & & \end{array}$$

Roesser Model

$$D = \mathbb{R}[\sigma_i, \sigma_j], \quad R = \begin{pmatrix} \sigma_i - 2 & -3 \\ -4 & \sigma_j - 5 \end{pmatrix} \in D^{2\times 2}$$

$$\begin{array}{ccl} D^{1\times 2} & \xrightarrow{.R} & D^{1\times 2} \\ (\lambda_1 \quad \lambda_2) & \longmapsto & (\lambda_1 \quad \lambda_2)\, R = (\lambda_1\,(\sigma_i - 2) + \lambda_2\,(-4) \quad \lambda_1\,(-3) + \lambda_2\,(\sigma_j - 5)) \end{array}$$

$\rightsquigarrow$ Associated left D-module $M = D^{1\times 2}/D^{1\times 2}\, R$

$$\begin{array}{ccc} D^{1\times 2} & \xrightarrow{\pi} & M \\ \delta = (\delta_1 \quad \delta_2) & \longmapsto & \pi(\delta) \end{array}$$

where $\pi(\delta)$ is the residue class of δ in M

$$\pi(\delta) = \pi(\delta') \Longleftrightarrow \exists\, \lambda \in D^{1\times 2};\ \delta = \delta' + \lambda\, R$$

In particular, if $\delta = \lambda\, R$, then $\pi(\delta) = \pi(0) = 0$

Roesser Model

◇ $f_1 = (1 \quad 0)$, $f_2 = (0 \quad 1)$ standard basis of $D^{1\times 2}$

◇ $y_1 = \pi(f_1)$, $y_2 = \pi(f_2)$ are generators of M: indeed $m \in M$

$$m = \pi(\delta) = \pi(\delta_1\, f_1 + \delta_2\, f_2) = \delta_1\, \pi(f_1) + \delta_2\, \pi(f_2) = \delta_1\, y_1 + \delta_2\, y_2$$

◇ These generators satisfy D-linear relations:

$$\begin{array}{rcl} (\sigma_i - 2)\, y_1 + (-3)\, y_2 & = & (\sigma_i - 2)\, \pi(f_1) - 3\, \pi_R(f_2) \\ & = & \pi\left((\sigma_i - 2)\, f_1 - 3\, f_2\right) \\ & = & \pi\left((\sigma_i - 2 \quad -3)\right) \\ & = & \pi((1 \quad 0)\, R) \\ & = & 0 \end{array}$$

Similarly, $\quad (-4)\, y_1 + (\sigma_j - 5)\, y_2 \quad = \quad \pi_R((0 \quad 1)\, R) = 0$

$\rightsquigarrow$ If $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, then it yields $R\, y = 0$

Key point of the algebraic analysis approach

⋄ D ring of functional operators, $R \in D^{q \times p}$

⋄ To the matrix R we associate the left D-module:

$$M = D^{1 \times p}/(D^{1 \times q}\, R)$$

⋄ For a functional space $\mathcal{F}$, consider the linear system (behavior)

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$$

Malgrange's remark:

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}) = \{f : M \to \mathcal{F},\ f \text{ is left } D\text{-linear}\}$$

Algebraic analysis approach

1. A linear system is defined by a matrix R with coefficients in a ring D of functional operators:
$$R\,y = 0 \quad (\star)$$
2. To $(\star)$ we associate a left D-module M (finitely presented)

   Malgrange's remark:
$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}) = \{f : M \to \mathcal{F},\ f \text{ is left } D\text{-linear}\}$$
3. There exists a dictionary between the properties of $(\star)$ and M
4. Homological algebra allows to check the properties of M
5. Effective algebra gives algorithms $\rightsquigarrow$ implementation

Dictionary

Algebraic prop. of M	**Structural prop. of** $\ker_{\mathcal{F}}(R.)$
Torsion	Poles/zeros classifications
With torsion	Existence of autonomous elements
Torsion free	No autonomous elements, Controllability, Parametrizability, π-flatness
Reflexive	Filter identification
Projective	Internal stabilizability, Bézout identities, Stabilizing controllers
Free	Flatness, Doubly coprime factorization, Poles placement, Youla-Kucera parametrization

◇ The algebraic properties of M can be checked using the Maple package OreModules developed by *Chyzak, Quadrat, Robertz*

Equivalence in this framework

◇ Definition: two linear systems are equivalent if their associated D-modules are isomorphic.

→ Study morphisms and isomorphisms between D-modules.

1. First step: (homo)morphism
2. Second step: isomorphism

◇ Studied from a computational point of view by *Cluzeau, Quadrat*

⇝ Maple package OreMorphisms

Morphisms between finitely presented D-modules

$\diamond$ Let $M = D^{1\times p}/(D^{1\times q}\,R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\,R')$

$$\begin{array}{ccccccc}
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle .Q} & & \downarrow{\scriptstyle .P} & & \downarrow{\scriptstyle f} & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0
\end{array}$$

$\diamond$ $\exists\, f \in \hom_D(M, M') \Longleftrightarrow \exists\, P \in D^{p\times p'},\ Q \in D^{q\times q'}$ such that:

$$R\,P = Q\,R'$$

$\diamond$ $f \in \hom_D(M, M')$ is defined by:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\,P)$$

$\diamond$ The matrix P sends solutions of $R'\,\eta' = 0$ to solutions of $R\,\eta = 0$

Explicit characterization of an isomorphism

⋄ $M = D^{1\times p}/(D^{1\times q}\, R)$, $M' = D^{1\times p'}/(D^{1\times q'}\, R')$

1. f is a morphism from M to M' iff there exist P and Q such that:

$$R\,P = Q\,R'$$

2. f is an isomorphism iff there exist P', Q', Z, and Z' such that:

$$R'\,P' = Q'\,R, \quad P\,P' + Z\,R = I_p, \quad P'\,P + Z'\,R' = I_{p'}$$

⋄ Changes of variables $\eta = P\,\eta'$, $\eta' = P'\,\eta \to R\,\eta = 0 \Leftrightarrow R'\,\eta' = 0$

Linear Repetitive Processes (LRP)

◇ State-space representation (*Rogers-Gałkowski-Owens'07*):

$$(\mathtt{LRP}) \left\{ \begin{array}{rcl} x_{k+1}(p+1) & = & \mathcal{A}\, x_{k+1}(p) + \mathcal{B}_0\, y_k(p) + \mathcal{B}\, u_{k+1}(p) \\ y_{k+1}(p) & = & \mathcal{C}\, x_{k+1}(p) + \mathcal{D}_0\, y_k(p) + \mathcal{D}\, u_{k+1}(p) \end{array} \right.$$

◇ Here $k \geq 0$ unbounded but in general $0 \leq p \leq \alpha$ for $\alpha \in \mathbb{N}^*$

$\rightarrow$ Differences with Roesser models (*Paszke'05, Rogers-Gałkowski-Owens'07, Boudellioua-Gałkowski-Rogers'17*)

$\rightarrow$ Here, we formally consider p unbounded as for the study of *stability along the pass* (*Sulikowski-Gałkowski-Rogers-Owens'04*)

◇ Previous approach: such a LRP is always equivalent to a Roesser model!

Equivalence with a Roesser model

◇ Theorem (*Bachelier-Cluzeau*)
(LRP) is equivalent to the Roesser model (R) with:

$$A = \left(\begin{array}{ccc|c} 0 & \mathcal{B}_0 & 0 & \mathcal{B}_0\,\mathcal{C} \\ 0 & \mathcal{D}_0 & 0 & \mathcal{D}_0\,\mathcal{C} \\ 0 & 0 & 0 & 0 \\ \hline I_{d_x} & 0 & 0 & \mathcal{A} \end{array}\right), B = \left(\begin{array}{c} \mathcal{B} \\ \mathcal{D} \\ I_{d_u} \\ \hline 0 \end{array}\right), C = \left(\begin{array}{ccc|c} 0 & I_{d_y} & 0 & \mathcal{C} \end{array}\right), D = 0.$$

◇ This is not unique. For instance, another choice is:

$$A = \left(\begin{array}{cc|cc} 0 & \mathcal{B}_0 & \mathcal{B}_0\,\mathcal{C} & \mathcal{B}_0\,\mathcal{D} \\ 0 & \mathcal{D}_0 & \mathcal{D}_0\,\mathcal{C} & \mathcal{D}_0\,\mathcal{D} \\ \hline I_{d_x} & 0 & \mathcal{A} & \mathcal{B} \\ 0 & 0 & 0 & 0 \end{array}\right), B = \left(\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ I_{d_u} \end{array}\right), C = \left(\begin{array}{cc|cc} 0 & I_{d_y} & \mathcal{C} & \mathcal{D} \end{array}\right), D = 0.$$

◇ **Example with Maple**

Some stability notions

◇ Structural stability: let $\mathcal{S} = \left\{(z_1, z_2) \in \overline{\mathbb{C}}^2 \mid \forall i = 1, 2, |z_i| \geq 1\right\}$

- A LRP is *structurally stable* if

$$\forall (\lambda_1, \lambda_2) \in \mathcal{S}, \quad \det \begin{pmatrix} \lambda_1 \, \lambda_2 \, I_{d_x} - \lambda_1 \, \mathcal{A} & -\mathcal{B}_0 \\ -\lambda_1 \, \mathcal{C} & \lambda_1 \, I_{d_y} - \mathcal{D}_0 \end{pmatrix} \neq 0$$

- A Roesser model is *structurally stable* if

$$\forall (\lambda_1, \lambda_2) \in \mathcal{S}, \quad \det \begin{pmatrix} \lambda_1 \, I_{d_h} - A_{11} & -A_{12} \\ -A_{21} & \lambda_2 \, I_{d_v} - A_{22} \end{pmatrix} \neq 0$$

◇ Stability along the pass A LRP is said *stable along the pass* if

i. for all $\lambda \in \overline{\mathbb{S}}$, $\det(\lambda I_{d_y} - \mathcal{D}_0) \neq 0$ (stability from pass to pass)

ii. for all $\lambda \in \overline{\mathbb{S}}$, $\det(\lambda I_{d_x} - \mathcal{A}) \neq 0$

iii. for all $\lambda \in \partial\overline{\mathbb{S}}$, $\det(G(\lambda)) \neq 0$, where $G(\lambda) = \mathcal{C}(\lambda I_{d_x} - \mathcal{A})^{-1}\mathcal{B}_0 + \mathcal{D}_0$ and $\partial\overline{\mathbb{S}} = \{z \in \overline{\mathbb{C}}, |z| = 1\}$ denotes the unit circle

Application to stability/stabilization

◇ a LRP is stable along the pass iff it is structurally stable

◇ Structural stability is preserved through the equivalence transformation!

◇ Application: stabilization (along the pass) of a given LRP:

1. Transform the LRP into an equivalent Roesser model
2. Compute a stabilizing state feedback control law for the Roesser model (*Bachelier et al*)
3. Transform this law through the equivalence to a stabilizing state feedback control law for the LRP model

Example

◇ Consider the LRP given by

$$\left(\begin{array}{c|c|c} \mathcal{A} & \mathcal{B}_0 & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D}_0 & \mathcal{D} \end{array}\right) = \left(\begin{array}{cc|c|c} 0.2059 & 0.4911 & 0.0144 & 0.4983 \\ 0.5398 & 0.7125 & 0 & 0.9597 \\ \hline 0.1626 & 0.1189 & 0.8990 & 0.3403 \end{array}\right).$$

It is not structurally stable and thus not stable along the pass

◇ We consider the equivalent Roesser model with matrices:

$$A = \left(\begin{array}{cc|cc} 0 & \mathcal{B}_0 & \mathcal{B}_0\,\mathcal{C} & \mathcal{B}_0\,\mathcal{D} \\ 0 & \mathcal{D}_0 & \mathcal{D}_0\,\mathcal{C} & \mathcal{D}_0\,\mathcal{D} \\ \hline I_2 & 0 & \mathcal{A} & \mathcal{B} \\ 0 & 0 & 0 & 0 \end{array}\right), B = \left(\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 1 \end{array}\right), C = \left(\begin{array}{cc|cc} 0 & 1 & \mathcal{C} & \mathcal{D} \end{array}\right), D = 0.$$

We know that it is not structurally stable

Example

◇ Using the work of *Bachelier et al*, one can compute (resolution of a LMI) a stabilizing state feedback control law for the Roesser model:

$$u' = K' \begin{pmatrix} x^h \\ x^v \end{pmatrix}, \quad K' = \begin{pmatrix} -0.3418 & -0.4437 & -0.0029 & -0.3207 & -0.4861 & -0.6175 \end{pmatrix}$$

◇ Using the explicit equivalence transformation, we deduce the stabilizing state feedback control law for the original LRP:

$$u_k(p+1) = -\begin{pmatrix} 0.3418 & 0.4437 \end{pmatrix} x_k(p+1) - \begin{pmatrix} 0.01034258 & 0.00176096 \end{pmatrix} x_k(p) + 0.0029\, y_k(p) - 0.0203753\, u_k(p)$$

◇ The induced closed-loop model obeys to the structure of the LRP:

$$\begin{cases} X_{k+1}(p+1) = \mathbf{A}\, X_{k+1}(p) + \mathbf{B_0}\, y_k(p) \\ y_{k+1}(p) = \mathbf{C}\, X_{k+1}(p) + \mathbf{D_0}\, y_k(p) \end{cases}$$

$$\mathbf{A} = \begin{pmatrix} 0.2059 & 0.4911 & 0.4983 \\ 0.5398 & 0.7125 & 0.9597 \\ -0.31975692 & -0.48541038 & -0.61552626 \end{pmatrix}, \mathbf{B_0} = \begin{pmatrix} 0.0144 \\ 0 \\ -0.00231482 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0.1626 & 0.1189 & 0.3403 \end{pmatrix}, \mathbf{D_0} = 0.899$$

◇ The latter LRP is structurally stable and thus stable along the pass!

Thank you for your attention!